Math 656 • FINAL EXAM • May 9, 2016

1) **(10pts)** Find **all values** of $\cosh^{-1}(2i)$, and plot them as point in the complex plane (hint: convert to a quadratic equation for e^z)

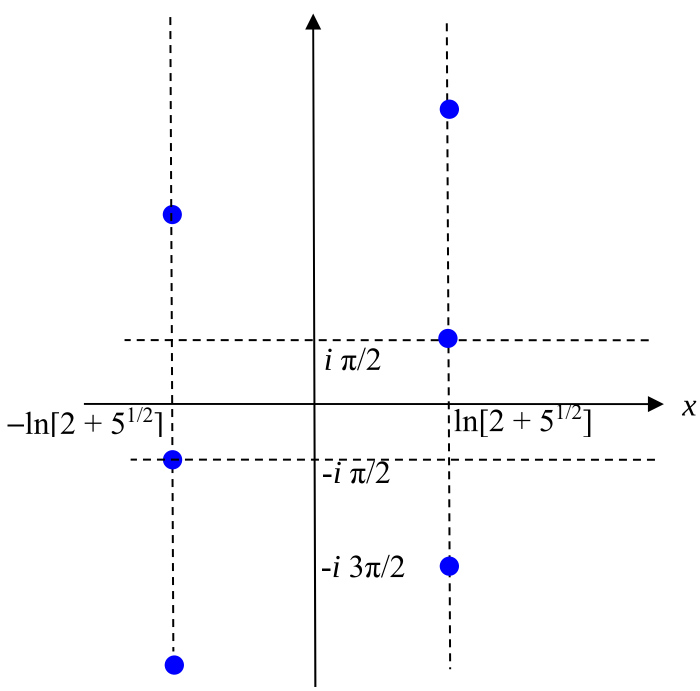
$$\cosh z = 2i \quad \Rightarrow \quad \frac{e^z + e^{-z}}{2} = 2i \quad \Rightarrow \quad e^z + e^{-z} - 4i = 0 \;\Big|\times e^z \quad \Rightarrow \quad \underbrace{e^{2z}}_{s^2} - 4i\underbrace{e^z}_{s} + 1 = 0$$

$$\Rightarrow \; s^2 - 4i\,s + 1 = 0 \;\Rightarrow\; s_{1,2} = \frac{4i + (-16-4)^{1/2}}{2} = i\left(2 \pm \sqrt{5}\,\right) \quad \Rightarrow \quad z = \log\left(i\left(2 \pm \sqrt{5}\,\right)\right)$$

Note that $2 - \sqrt{5} < 0$ so $2 - \sqrt{5} = e^{i\pi}\left(\sqrt{5} - 2\right)$

$$z = \begin{cases} \log\left(i\left(2+\sqrt{5}\,\right)\right) = \ln\left(\sqrt{5}+2\right) + i\left(\dfrac{\pi}{2} + 2\pi n\right) \\ \log\left(i\left(2-\sqrt{5}\,\right)\right) = \ln\left(\sqrt{5}-2\right) - i\left(\dfrac{\pi}{2} + 2\pi n\right) \end{cases} \Rightarrow \boxed{z = \cosh^{-1}(2i) = \pm\left[\ln\left(\sqrt{5}+2\right) + \pi\left(\frac{1}{2} + 2n\right)\right] \quad n \in \mathbb{Z}}$$

In the last step we used the fact that $\ln\left(\sqrt{5}-2\right) = -\ln\left(\sqrt{5}+2\right)$ because $\left(\sqrt{5}-2\right)\left(\sqrt{5}+2\right) = 1$



2) (24pts) Describe all singularities of the integrand inside the integration contour, and calculate each integral (use any method you like). Each integration contour is a circle of specified radius

(a) $\oint\limits_{|z|=3} \frac{dz}{(z^2+1)^2}$ (b) $\oint\limits_{|z|=5} \frac{dz}{z^2 \cos z}$ (c) $\oint\limits_{|z|=3} \frac{dz}{\bar{z}}$ (d) $\oint\limits_{|z|=1} \exp\left(\frac{1}{z}+z\right)dz$

(a) Two poles of order 2 inside integraiton contour; use the standard pole formula for residues (the answer will be be zero since the integrand is even):

$$\oint\limits_{|z|=3} \frac{dz}{(z^2+1)^2} = \oint\limits_{|z|=3} \frac{dz}{(z+i)^2(z-i)^2} = 2\pi i\left[\operatorname{Res}\left(\frac{1/(z+i)^2}{(z-i)^2};\, i\right) + \operatorname{Res}\left(\frac{1/(z-i)^2}{(z+i)^2};\, -i\right)\right]$$

$$= 2\pi i\left[\frac{d}{dz}\frac{1}{(z+i)^2}\bigg|_{z=+i} + \frac{d}{dz}\frac{1}{(z-i)^2}\bigg|_{z=-i}\right] = 2\pi i\left[-\frac{2}{(2i)^3} - \frac{2}{(-2i)^3}\right] = \boxed{0}$$

(b) Two simple poles at $z = \pm\frac{\pi}{2}$ and a pole of order 2 at $z = 0$ (the answer has to be zero since the integrand is even):

$$\oint\limits_{|z|=5} \frac{dz}{z^2 \cos z} = 2\pi i\left[\operatorname{Res}\left(\frac{1/z^2}{\cos z};\, \frac{\pi}{2}\right) + \operatorname{Res}\left(\ldots;\, -\frac{\pi}{2}\right) + \operatorname{Res}\left(\ldots;\, \frac{3\pi}{2}\right) + \operatorname{Res}\left(\ldots;\, -\frac{3\pi}{2}\right) + \operatorname{Res}\left(\frac{1/\cos z}{z^2};\, 0\right)\right]$$

$$= 2\pi i\left[-\frac{1}{z^2 \sin z}\bigg|_{z=\frac{\pi}{2}} - \frac{1}{z^2 \sin z}\bigg|_{z=-\frac{\pi}{2}} - \frac{1}{z^2 \sin z}\bigg|_{z=\frac{3\pi}{2}} - \frac{1}{z^2 \sin z}\bigg|_{z=-\frac{3\pi}{2}} + \frac{d}{dz}\frac{1}{\cos z}\bigg|_{z=0}\right]$$

$$= 2\pi i\left[-\left(\frac{2}{\pi}\right)^2 + \left(-\frac{2}{\pi}\right)^2 - \left(\frac{2}{3\pi}\right)^2 + \left(-\frac{2}{3\pi}\right)^2 + \frac{\sin 0}{\cos^2 0}\right] = \boxed{0}$$

(c) Singular in the entire $\mathbb{C}$; use parametrization: $\oint\limits_{|z|=3} \frac{dz}{\bar{z}} = \left| \begin{array}{l} z = 3e^{i\theta} \\ 1/\bar{z} = e^{i\theta}/3 \\ dz = ie^{i\theta}d\theta \end{array} \right| = \int_0^{2\pi} e^{2i\theta}d\theta = \frac{\left[e^{2i\theta}\right]_0^{2\pi}}{2i} = \boxed{0}$

(d) Essential singularity at $z = 0$: $\oint\limits_{|z|=1} \exp\left(\frac{1}{z}+z\right)dz = \oint\limits_{|z|=1}\left[\left(1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\ldots\right)\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots\right)\right]dz$

$$= 2\pi i \operatorname{Res}(0) = 2\pi i\left(1+\frac{1}{2!}+\frac{1}{2!3!}+\frac{1}{3!4!}+\ldots\right) = \boxed{2\pi i\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}}$$

3) (10pts) Find the first **two** dominant terms in the series expansion of $f(z)=\dfrac{\cos z-1}{z^2\left(e^z-1\right)}$ near z = 0, and use your result to classify the singularity at z=0. What is the residue of this function at z=0? What would be the domain of convergence of the corresponding full series? Finally, classify the singularity of this function at z=∞, as well.

$$f(z)=\frac{\cos z-1}{z^2\left(e^z-1\right)}=\frac{1-\frac{z^2}{2}+\frac{z^4}{4!}+O\left(z^6\right)-1}{z^2\left(1+z+\frac{z^2}{2}+O\left(z^3\right)-1\right)}=\frac{-\frac{z^2}{2}+\frac{z^4}{4!}+O\left(z^6\right)}{z^2\left(z+\frac{z^2}{2}+O\left(z^3\right)\right)}=\frac{-\frac{z^2}{2}\left(1-\frac{2z^3}{4!}+O\left(z^5\right)\right)}{z^3\left(1+\frac{z}{2}+O\left(z^2\right)\right)}$$

$$=-\frac{1}{2z}\frac{\left(1-\frac{2z^3}{4!}+O\left(z^5\right)\right)}{\left(1+\frac{z}{2}+O\left(z^2\right)\right)}=-\frac{1}{2z}\left(1-\frac{2z^3}{4!}+O\left(z^5\right)\right)\left(1-\frac{z}{2}+O\left(z^2\right)\right)=-\frac{1}{2z}\left(1-\frac{z}{2}+O\left(z^2\right)\right)=\boxed{-\frac{1}{2z}+\frac{1}{4}+O(z)}$$

- Residue at z=0 (which is a **simple pole**): $\boxed{C_{-1}=-\frac{1}{2}}$

- Domain of convergence: closest singulatiry is at z=2π: $\boxed{\text{converges in } 0<|z|<2\pi}$

- Singularity at z=∞: $\boxed{\text{cluster point, since it is a limit of simple poles } z_n=2\pi n}$

Note that is is **impossible** to find a series around $w_o=0$ where $w=\frac{1}{z}$, even though many of you tried:

$$\frac{1}{e^z-1}=\frac{1}{e^{1/w}-1}=\frac{1}{\frac{1}{w}+\frac{1}{2w^2}+\frac{1}{3!w^3}+\ldots}=\frac{w}{1+\frac{1}{2w}+\frac{1}{3!w^2}+\ldots}$$

This result is true, but there is **no way** to continue after this step: this **cannot** be transformed into a power series in w or $1/w$

4) (10pts) Sketch the domain of convergence of the Laurent series $\sum_{k=0}^{\infty}\left[\frac{(2i+z)^k}{e^{2k}}+\frac{3^k}{k!\,(2i+z)^{2k}}\right]$, and write down the expression for its sum. What are the singularities of this sum (which represents the analytic extension of this series)? **Hint:** this is a very straightforward problem: notice a combination of standard series of familiar elementary functions.

This is a Laurent series centered at $z_o = -2i$, so the series converges in some **annulus ("ring")** $0 \le r < |z - z_o| < R$

Now, the first term is a geometric series, and the second term is a Taylor series of an exponential function:

$$\sum_{k=0}^{\infty}\left[\frac{(2i+z)^k}{e^{2k}}+\frac{3^k}{k!\,(2i+z)^{2k}}\right]=\sum_{k=0}^{\infty}\left(\frac{z+2i}{e^2}\right)^k+\sum_{k=0}^{\infty}\frac{\left(3/(z+2i)^2\right)^k}{k!}$$

$$=\frac{1}{1-\frac{z+2i}{e^2}}+\exp\left(\frac{3}{(z+2i)^2}\right)=\boxed{\frac{e^2}{e^2-2i-z}+\exp\left(\frac{3}{(z+2i)^2}\right)}$$

This sum has an essential singularity at $z_o=-2i$, and a pole at $z-z_o = e^2$, so the convergence annulus of its series expansion is $\boxed{0<|z+2i|<e^2}$ (this can also be determined using the ratio test)

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5) (16pts) Calculate **two** of the following integrals. Explain each step briefly but fully. If you choose (c), use an "indented" contour. Make sure to obtain a real answer in each problem!

$$(a)\quad \int_0^{2\pi}\frac{d\theta}{3+2\cos\theta}=\left|\begin{array}{l} z=e^{i\theta}\\ dz=iz\,d\theta\\ \cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)\end{array}\right|=\oint_{|z|=1}\frac{dz}{zi\left(3+z+\frac{1}{z}\right)}=-i\oint_{|z|=1}\frac{dz}{z^2+3z+z}=-i\oint_{|z|=1}\frac{dz}{(z-z_1)(z-z_2)}$$

where $z_{1,2}=\frac{-3\pm\sqrt{5}}{2}$; note that only $z_1=\frac{-3+\sqrt{5}}{2}$ is inside the unit circle, therefore:

$$-i\oint_{|z|=1}\frac{dz}{(z-z_1)(z-z_2)}=-i(2\pi i)\,\mathrm{Res}(z_1)=\frac{2\pi}{2z_1+3}=\boxed{\frac{2\pi}{\sqrt{5}}}$$

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$$(b)\ \int_{-\infty}^{+\infty}\frac{x^5\sin(2x)\,dx}{1+x^6}=\mathrm{Im}\left[\int_{-\infty}^{+\infty}\frac{x^5e^{2ix}\,dx}{1+x^6}\right]$$ Three simple poles in upper half-plane:

$$z_{1,2,3}=(-1)^{1/6}=(e^{i\pi+i2\pi k})^{1/6}=\{e^{i\pi/6},e^{i3\pi/6},e^{i5\pi/6}\}=\left\{\frac{\pm\sqrt{3}+i}{2};\ i\right\}$$

Closing via semi-circle C_R in the upper half-plane:

$$\oint_C\frac{e^{i2z}\,dz}{a^4+z^4}dz=\int_{-R}^{R}\frac{e^{i2z}\,dx}{a^4+x^4}dx+\int_{C_R}\frac{e^{i2z}\,dz}{a^4+z^4}=2\pi i\left\{\mathrm{Res}\left(\frac{\sqrt{3}+i}{2}\right)+\mathrm{Res}\left(\frac{-\sqrt{3}+i}{2}\right)+\mathrm{Res}(i)\right\}$$

$$=2\pi i\left(\frac{z_1^5e^{i2z_1}}{5z_1^5}+\frac{z_2^5e^{i2z_2}}{5z_2^5}+\frac{z_3^5e^{i2z_3}}{5z_3^5}\right)=\frac{2i\pi}{5}\left[e^{i2z_1}+e^{i2z_2}+e^{i2z_3}\right]=\frac{2i\pi}{5}\left[e^{i\sqrt{3}-1}+e^{-i\sqrt{3}-1}+e^{-2}\right]=\boxed{\frac{2i\pi}{5}\left[2e^{-1}\cos\sqrt{3}+e^{-2}\right]}$$

The integral over C_R can be shown to approach zero as $R\to\infty$, by the Jordan's Lemma:

$$\left|\int_{C_R}\frac{z^5e^{2iz}\,dz}{1+z^6}\right|\le\frac{\pi}{2}\frac{R^5}{R^6-1}\to\frac{\pi/2}{R}\to 0\quad\text{as}\quad R\to\infty\Rightarrow$$ Taking imaginary part in the limit $R\to\infty$:

$$\Rightarrow\boxed{\int_{-\infty}^{\infty}\frac{x^5\sin(2x)\,dx}{1+x^6}=\frac{2\pi}{5e^2}\left[2e\cos\sqrt{3}+1\right]}$$

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$$(c)\ \int_{-\infty}^{+\infty}\frac{\cos x-\cos(2x)}{x^2}dx=\underbrace{CPV\int_{-\infty}^{+\infty}\frac{e^{ix}-e^{i2x}}{x^2}dx}_{\text{Cauchy Principal Value}}$$ (imaginary part =0: it's an even function)

Consider a semi-circular contour closed in the upper half-plane and indented at the origin:

$$\oint_{\Gamma_R}\frac{e^{iz}-e^{i2z}}{z^2}dz=\underbrace{\int_{C_\varepsilon^-}\frac{e^{iz}-e^{i2z}}{z^2}dz}_{-\frac{1}{2}2\pi i\,\mathrm{Res}(0)}+\underbrace{\int_{C_R^+}\frac{e^{iz}-e^{i2z}}{z^2}dz}_{|...|\le\frac{\pi R}{R^2}}+\int_{-R}^{-\varepsilon}\frac{e^{ix}-e^{i2x}}{x^2}dx+\int_{\varepsilon}^{R}\frac{e^{ix}-e^{i2x}}{x^2}dx=0$$

Taking the limit $\varepsilon\to 0$ and $R\to\infty$ (note that Jordan's Lemma is not required here), we obtain

$$CPV\int_{-\infty}^{+\infty}\frac{e^{ix}-e^{i2x}}{x^2}dx=-\pi i\ \mathrm{Res}\left(\frac{e^{iz}-e^{i2z}}{z^2},\ 0\right)=-\pi i\ \underbrace{\mathrm{Res}\left(\frac{1+iz+O(z^2)-\left(1+i2z+O(z^2)\right)}{z^2},\ 0\right)}_{=-i}=-\pi i\cdot i=\boxed{\pi}$$

6) **(10pts)** Use Rouche's Theorem to find an annulus/ring with integer radii, $n<|z|<n+1$ $\left(n\in\mathbb{Z}_+\right)$, containing all roots of polynomial $f(z)=z^3+z^2+40$

Trying a couple values of *n* easilly yields the result: $3<|z|<4$

=================== **You may drop one problem out of the last three** ===================

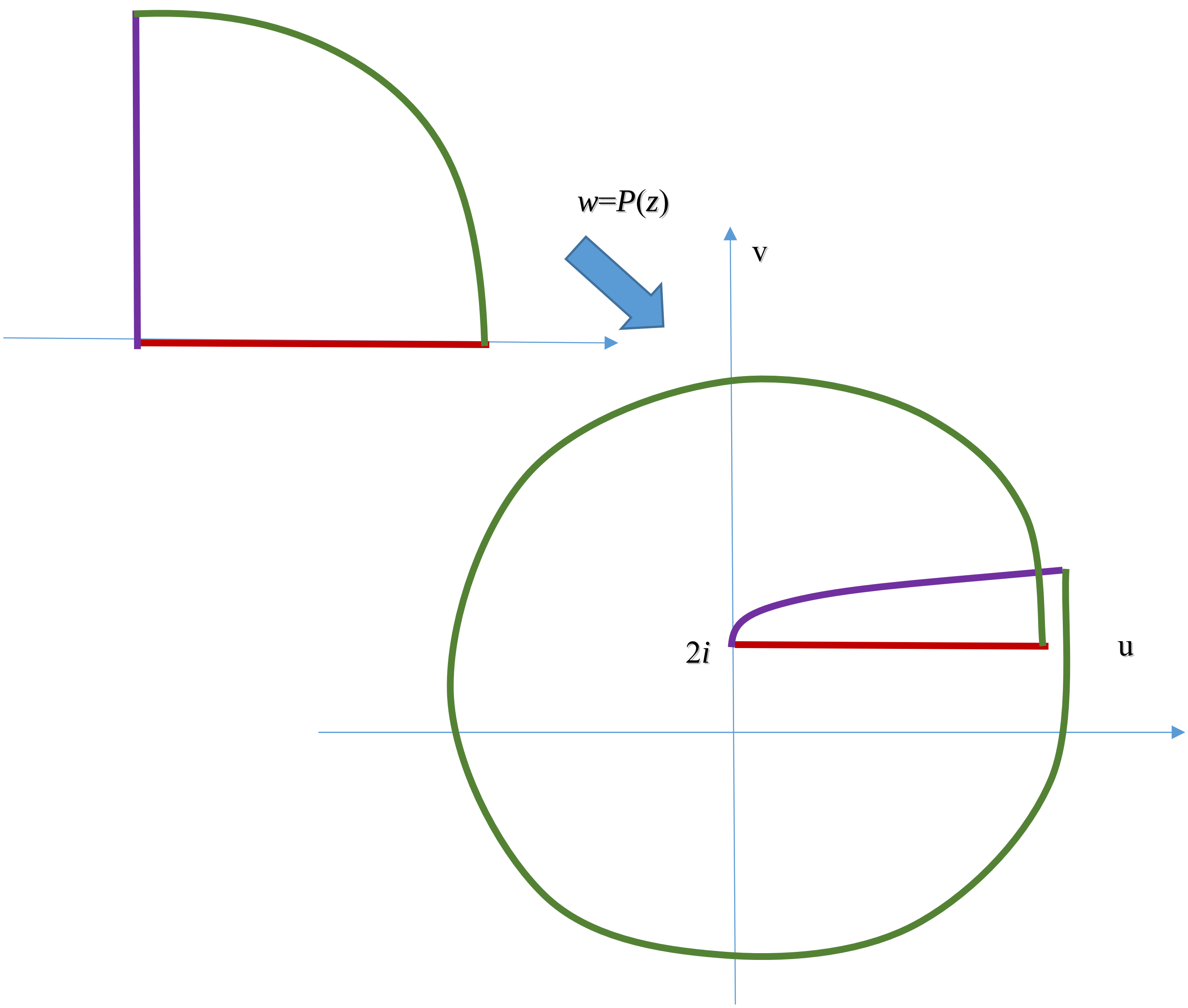
7) **(10pts)** Use the Argument Principle to find the number of roots of $f(z) = 2i + z + z^4$ lying in the first quadrant. To do this, sketch the mapping of the relevant quarter-circular sector boundary (it's quite straightforward).

Consider $P(z) = 2i + z + z^4$ as a map from (x, y) to (u, v)

Map the three boundaries of the first quadrant; only the map of the imaginary axis requires a little thought:

$$P(z = iy) = 2i + 2iy + y^4 \equiv u + iv \;\Rightarrow\; \begin{cases} u = +y^4 & \text{(monotonically increasing)} \\ v = 2 + 2y & \text{(monotonically increasing)} \end{cases}$$

Winding number of $w = P(z)\big|_{z\in C}$ around $w = 0$ equals one $\Rightarrow$ **One** root in the 1st quadrant



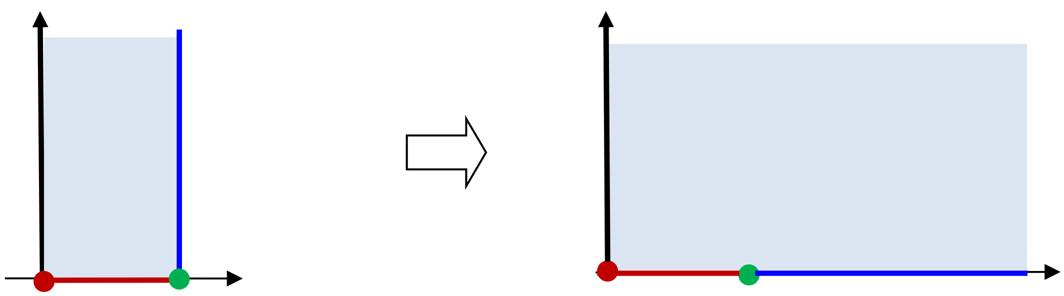
8) (10pts) What is the image of the domain $\left\{ \mathrm{Re}(z) \in \left[0,\ \frac{\pi}{2}\right],\ \mathrm{Im}(z) \in [0,\ +\infty] \right\}$ under the mapping $w = \sin z$?

Hint: consider separately the map of each boundary, and the map of any point or curve within this domain. You may use the Cartesian decomposition $\sin z = \sin x \cosh y + i\ \cos x \sinh y$. Note that a map does not preserve angles (is not conformal) wherever $f'(z) = 0$.

This semi-infinite strip is mapped into entire first quadrant (easily seen by mapping the two corners and the three boundaries:

$$\begin{cases} 1.\ \ z = iy:\ \ \sin(iy) = i \sinh y \ \Rightarrow\ \text{Left vertical edge maps to infinite vertical line} \\ 2.\ \ z = \frac{\pi}{2} + iy:\ \sin\left(\frac{\pi}{2} + iy\right) = \cos(iy) = \cosh y \ \Rightarrow \text{Right vertical edge maps to horizontal line } [1, +\infty) \\ 3.\ \ z = x \in \left[0,\ \frac{\pi}{2}\right] \Rightarrow \sin x \in [0,\ 1]\ \ \text{Real segement maps to real segment} \end{cases}$$

Note that the right angle at z=π/2 is doubled, since this point is a simple zero of this function



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9) (10pts) Suppose *f*(*z*) and *g*(*z*) are entire functions, and that $|f(z)| \leq 10\,|g(z)|$ in the entire complex plane. Is it true that $f(z) = \lambda g(z)$ for all *z*, where λ is a constant? If true, explain carefully, using any theorem learned in this course. If not true, give a counterexample. Note that *f*(*z*) and *g*(*z*) may have zeros in the complex plane.

This statement is true, and immediately follows from the Liouville Theorem for $h(z) = \dfrac{f(z)}{g(z)}$.

However, one has to prove that $h(z)$ is an entire function, despite the fact that $g(z)$ may have zeros.
Since $|f(z)| \leq 10\,|g(z)|$ near each zero, $f(z)$ has a zero of orden $n \geq m$ for each zero of order m of function $g(z)$.
Thus, all zeros of $g(z)$ are also zeros of $f(z)$, and are removable singularities of $h(z)$. Thus, the Liouville theorem can be applied to $h(z)$, and since $|h(z)| \leq 10$ (bounded!), we have $h(z) = const = \lambda \Rightarrow f(z) = \lambda g(z)$